Drugi parcijalni pismeni ispit iz predmeta **Analiza III**, 18.01.2013.

1. Izračunati zapreminu tijela koju ravan $z = x + y$ odsijeca od paraboloida $z = x^2 + y^2$.

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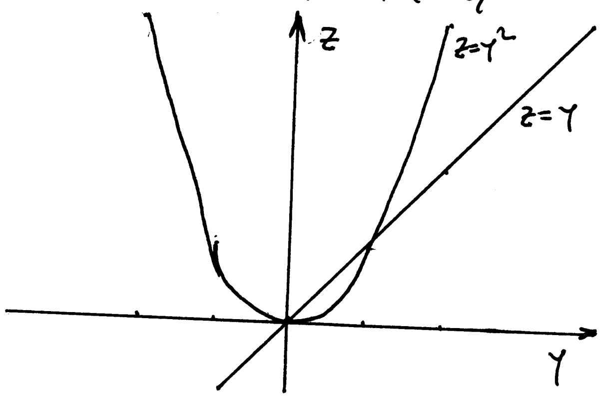
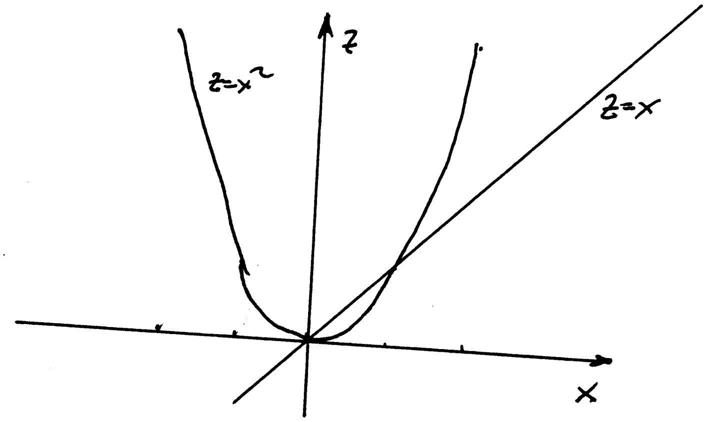
(#) Izračunati zapreminu tijela koju ravan $z=x+y$ odsijeca od paraboloida $z=x^2+y^2$.

Rj. Pogledajmo kako izgleda presjek datih površina sa yOz i xOz ravni





Na osnovu ove dvije slike pokušajte skicirati tijelo u prostoru!

$$V=\iiint\limits_{\Omega} dx\,dy\,dz=\iint\limits_{D} dx\,dy\int\limits_{x^2+y^2}^{x+y} dz=\iint\limits_{D}\left(x+y-(x^2+y^2)\right)dx\,dy \overset{(*)}{=}$$

gdje je D ortogonalna projekcija datog tijela na xOy ravan.

Projekciju presjeka tijela određujemo na sljedeći način

$$\begin{aligned} z&=x+y\\ z&=x^2+y^2\\ \hline \end{aligned}$$

$$x+y=x^2+y^2 \implies x^2-x+y^2-y=0$$

$$x^2-2\cdot x\cdot\frac{1}{2}+\frac{1}{4}+y^2-2\cdot y\cdot\frac{1}{2}+\frac{1}{4}=\frac{1}{2}$$

$$D:\ \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$$

Ako uvedemo polarne koordinate $x=\frac{1}{2}+r\cos\varphi$, $y=\frac{1}{2}+r\sin\varphi$, $dx\,dy=r\,dr\,d\varphi$

$D \xrightarrow{\text{transformire}} D'$

$$D' : \begin{cases} 0 \leq r \leq \frac{1}{\sqrt{2}} \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$\overset{(*)}{=} \iint_D (-1)(x^2 - x + y^2 - y)\,dx\,dy = (-1)\iint_D \left(\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2 - \frac{1}{2} \right) dx\,dy =$$

Primjetimo da je $x - \frac{1}{2} = r\cos\varphi$

$y - \frac{1}{2} = r\sin\varphi$

$$= (-1)\iint_{D'} \left(r^2 - \frac{1}{2}\right) r\,dr\,d\varphi = (-1)\int_0^{2\pi} d\varphi \int_0^{1/\sqrt{2}} \left(r^3 - \frac{1}{2}r\right) dr = \ldots = \frac{\pi}{8}$$

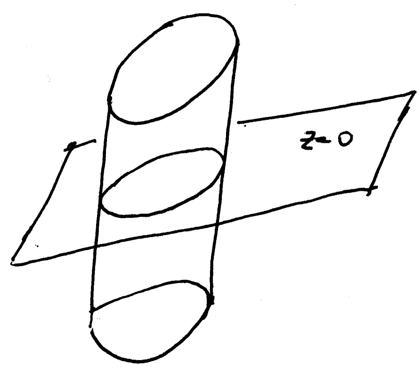
traženo rješenje

(#) Izračunati krivolinijski integral

$$I = \oint_C y\,dx + x^2\,dy$$

duž krive koja nastaje kao presjek ravni $z=0$ i cilindra $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}$ orjentisana u pozitivnom smjeru $(a \geq b > 0)$.

Rj. Za rješenje zadatka nije nam bitno gdje se cilindar nalazi u prostoru



$$z=0$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}$$

$$\frac{1}{a^2}(x^2 - ax) + \frac{1}{b^2}(y^2 - by) = 0$$

$$\frac{1}{a^2}\left(x^2 - 2\cdot x\cdot\frac{a}{2} + \frac{a^2}{4} - \frac{a^2}{4}\right) + \frac{1}{b^2}\left(y^2 - 2\cdot y\cdot\frac{b}{2} + \frac{b^2}{4} - \frac{b^2}{4}\right) = 0$$

$$\frac{1}{a^2}\left(x - \frac{a}{2}\right)^2 - \frac{1}{4} + \frac{1}{b^2}\left(y - \frac{b}{2}\right)^2 - \frac{1}{4} = 0$$

$$\frac{\left(x-\frac{a}{2}\right)^2}{a^2} + \frac{\left(y-\frac{b}{2}\right)^2}{b^2} = \frac{1}{2} \qquad /\cdot 2$$

$$\frac{\left(x-\frac{a}{2}\right)^2}{\frac{a^2}{2}} + \frac{\left(y-\frac{b}{2}\right)^2}{\frac{b^2}{2}} = 1$$

ovo je elipsa

Elipsu ćemo parametrizirati pomoću poopštenih polarnih koordinata

$$x = \frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi \qquad dx = -\frac{a}{\sqrt{2}}\sin\varphi\, d\varphi$$

$$y = \frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi \qquad dy = \frac{b}{\sqrt{2}}\cos\varphi\, d\varphi$$

$$0 \le \varphi < 2\pi$$

Sad nije teško izračunati dati krivolinijski integral

$$I = \oint_C y\,dx + x^2\,dy = \int_0^{2\pi}\left[\left(\frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi\right)\frac{-a}{\sqrt{2}}\sin\varphi + \left(\frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi\right)^2 \cdot \frac{b}{\sqrt{2}}\cos\varphi\right] d\varphi$$

$$= \frac{-a}{\sqrt{2}}\int_0^{2\pi}\left(\frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi\right)\sin\varphi\, d\varphi + \frac{b}{\sqrt{2}}\int_0^{2\pi}\left(\frac{a^2}{4} + \frac{a^2}{\sqrt{2}}\cos\varphi + \frac{a^2}{2}\cos^2\varphi\right)\cos\varphi\, d\varphi$$

$$= \frac{-ab}{2\sqrt{2}}\int_0^{2\pi}\sin\varphi\, d\varphi - \frac{ab}{2}\int_0^{2\pi}\underbrace{\sin^2\varphi}_{\frac{1}{2}(1-\cos 2\varphi)}\, d\varphi + \frac{a^2 b}{4\sqrt{2}}\int_0^{2\pi}\cos\varphi\, d\varphi + \frac{a^2 b}{2}\int_0^{2\pi}\underbrace{\cos^2\varphi}_{\frac{1}{2}(1+\cos 2\varphi)}\, d\varphi + \frac{a^2 b}{2\sqrt{2}}\int_0^{2\pi}\cos^3\varphi\, d\varphi$$

$$= \frac{-ab}{2\sqrt{2}}\underbrace{\cos\varphi\Big|_0^{2\pi}}_{=1-1=0} - \frac{ab}{2}\cdot\frac{1}{2}\left(\varphi\Big|_0^{2\pi} - \underbrace{\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}}_{=0}\right) + 0 + \frac{a^2 b}{2}\cdot\frac{1}{2}\left(\varphi\Big|_0^{2\pi} + \underbrace{\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}}_{=0}\right)$$

$$+ \frac{a^2 b}{2\sqrt{2}}\int_0^{2\pi}\cos^2\varphi\cos\varphi\, d\varphi =$$

$$1 = \sin^2\varphi + \cos^2\varphi$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$

$$1 - \cos 2\varphi = 2\sin^2\varphi$$

$$\sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi)$$

$$1 + \cos 2\varphi = 2\cos^2\varphi$$

$$= -\frac{ab\pi}{2} + \frac{a^2 b\pi}{2} + \frac{a^2 b}{2\sqrt{2}}\underbrace{\int_0^{2\pi}(1 - \sin^2\varphi)\, d(\sin\varphi)}_{=0}$$

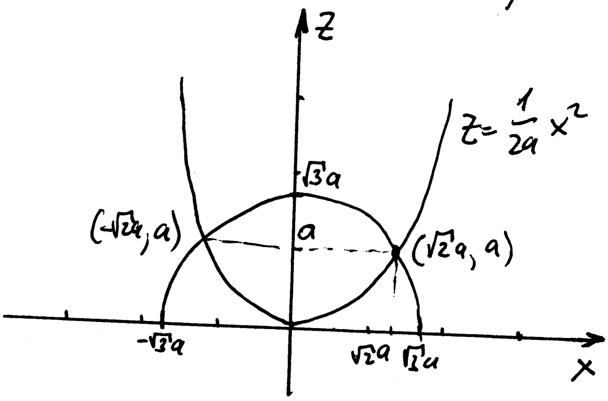
$$= \frac{\pi ab(-1+a)}{2} = \frac{ab\pi}{2}(a-1)$$

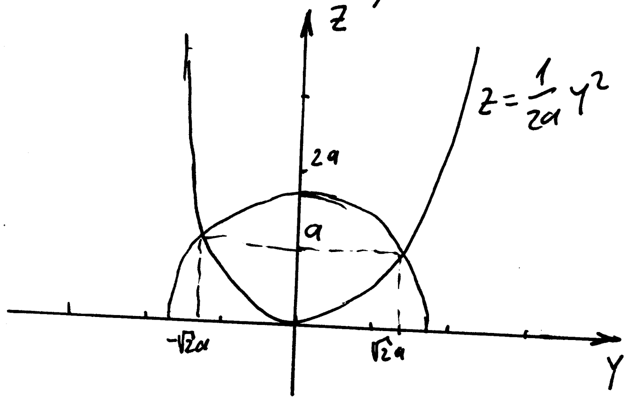
traženo rješenje

II način: Greenova formula ...

\# Izračunati površinu dijela lopte $x^2+y^2+z^2=3a^2$ koja se nalazi ispod parabole $x^2+y^2=2az$ a iznad xOy ravni.

Rj: Na osnovu skica presjeka datih površina sa xOz i yOz ravnima ćemo vidjeti kakva tijela su u pitanju.





$$x^2+z^2=3a^2$$
$$x^2=2az$$

$$z^2+2az-3a^2=0$$
$$D=4a^2+12a^2=16a^2$$
$$z_{1,2}=\frac{-2a\pm 4a}{2}$$
$$z_1=a \quad z_2=-3a$$

$$P=\iint_S dS$$ površinski integral prve vrste

$$z^2=3a^2-x^2-y^2$$
$$z=\pm\sqrt{3a^2-x^2-y^2}$$

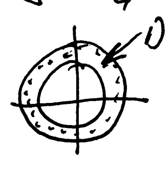
U našem slučaju S je $z=\sqrt{3a^2-x^2-y^2}$ i to dio ove površine koji se nalazi ispod parabole

$$P=\iint_S dS=\iint_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$z'_x=\frac{-2x}{2\sqrt{3a^2-x^2-y^2}}, \quad z'_y=\frac{-y}{\sqrt{3a^2-x^2-y^2}}$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{3a^2-x^2-y^2}+\frac{y^2}{3a^2-x^2-y^2}=\frac{3a^2}{3a^2-x^2-y^2}$$

$$P=\sqrt{3}a\iint_D\frac{dx\,dy}{\sqrt{3a^2-x^2-y^2}}$$ gdje je D projekcija površi S na xOy ravan. U našem slučaju



Uvedimo polarne koordinate

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$dx\,dy = r\,dr\,d\varphi$$

$$D \xrightarrow{\text{transformiše}} D' : \begin{cases} \sqrt{2}a \le r \le \sqrt{3}a \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$x^2 + y^2 = r^2$$

$$P = \sqrt{3}a \iint\limits_{D'} \frac{r\,dr\,d\varphi}{\sqrt{3a^2 - r^2}} = \sqrt{3}a \int_0^{2\pi} d\varphi \int_{\sqrt{2}a}^{\sqrt{3}a} \frac{r\,dr}{\sqrt{3a^2 - r^2}} = \left| \begin{array}{l} 3a^2 - r^2 = t^2 \\ -2r\,dr = 2t\,dt \\ r\Big|_{\sqrt{2}a}^{\sqrt{3}a} \Rightarrow t\Big|_a^0 \end{array} \right|$$

$$= \sqrt{3}a \int_0^{2\pi} d\varphi \int_0^a \frac{t\,dt}{t} = 2a^2\sqrt{3}\,\pi$$

traženo rešenje

2π a